Euler's partition theorem and the combinatorics of ℓ-sequences

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Inspiration

BME1 Mireille Bousquet-Mélou, and Kimmo Eriksson, *Lecture hall partitions*,
Ramanujan. J., 1:1, 1997, 101-111.

BME2 Mireille Bousquet-Mélou, and Kimmo Eriksson, *Lecture hall partitions II*,
Ramanujan. J., 1:2, 1997, 165-185.

Reference

SY07 C. D. S. and Ae Ja Yee, *Euler's partition theorem and the combinatorics of ℓ-sequences*,
JCTA, 2007, to appear, available online.

LS08 Nicholas Loehr, and C. D. S. *Notes on ℓ-nomials*, in preparation.

Overview

Euler's partition theorem

Overview

$1, \; 2, \;\; 3, \;\; \ldots$

ℓ-sequences

Euler's partition theorem

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$1, 2, 3, \ldots$

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The ℓ-Euler theorem

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Euler's partition theorem

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Lecture hall partitions

ℓ-Lecture hall partitions

Overview

$1, \; 2, \;\; 3, \;\; \ldots$
ℓ-sequences

Euler's partition theorem
The ℓ-Euler theorem

Lecture hall partitions
ℓ-Lecture hall partitions

Binomial coefficients
ℓ-nomial coefficients

Euler's Partition Theorem:

The number of partitions of an integer N into **odd** parts is equal to the number of partitions of N into **distinct** parts.

Example: $N = 8$

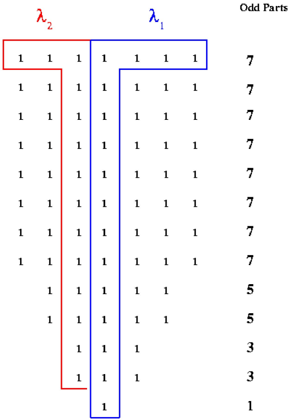
Odd parts:

(7,1) (5,3) (5,1,1,1) (3,3,1,1) (3,1,1,1,1,1) (1,1,1,1,1,1,1,1)

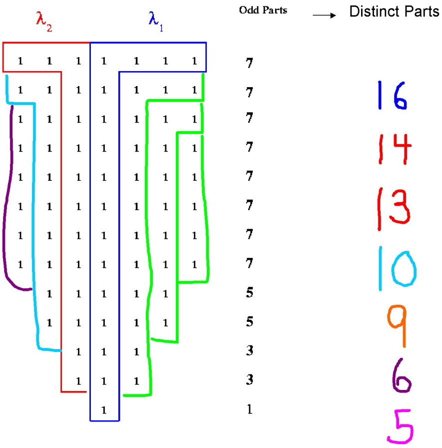
Distinct Parts:

(8) (7,1) (6,2) (5,3) (5,2,1) (4,3,1)

Sylvester's Bijection



Sylvester's Bijection



ℓ-sequences

For integer $\ell \geq 2$, define the sequence $\{a_n^{(\ell)}\}_{n\geq 1}$ by

$$a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)},$$

with initial conditions $a_1^{(\ell)} = 1$, $a_2^{(\ell)} = \ell$.

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with initial conditions $a_1^{(\ell)} = 1,\ a_2^{(\ell)} = \ell$.

$$\{a_n^{(3)}\} = 1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377, \ldots$$

$$\{a_n^{(2)}\} = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7, \ldots$$

ℓ-sequences

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$$\{a_n^{(3)}\} = 1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377, \ldots$$

$$\{a_n^{(2)}\} = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7, \ldots$$

(These are the (k, l) sequences in [BME2] with $k = l = \ell$.)

$\ell \geq 2$

The ℓ-Euler theorem [BME2]: The number of partitions of an integer N into parts from the set

$$\{a_0^{(\ell)} + a_1^{(\ell)},\ a_1^{(\ell)} + a_2^{(\ell)},\ a_2^{(\ell)} + a_3^{(\ell)},\ \ldots\}$$

is the same as the number of partitions of N in which the ratio of consecutive parts is greater than

$$c_\ell \;=\; \frac{\ell + \sqrt{\ell^2 - 4}}{2}$$

Proof: via *lecture hall partitions.*

$\ell = 2$

The ℓ-Euler theorem [BME2]: The number of partitions of an integer N into parts from the set

$$\{0+1,\ 1+2,\ 2+3,\ \ldots\} \ = \ \{1,\ 3,\ 5, \ldots\}$$

is the same as the number of partitions of N in which the ratio of consecutive parts is greater than

$$c_2 \ = \ \frac{2+\sqrt{2^2-4}}{2} \ = \ 1$$

$\ell = 3$

The ℓ-Euler theorem [BME2]: The number of partitions of an integer N into parts from the set

$$\{0+1,\ 1+3,\ 3+8,\ \ldots\} \ = \ \{1,\ 4,\ 11,\ 29, \ldots\}$$

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$$c_3 \ = \ \frac{3+\sqrt{3^2-4}}{2} \ = \ (3+\sqrt{5})/2$$

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Stanley: bijection?

$\Theta^{(\ell)}$: **Bijection for the ℓ-Euler Theorem** [SY07]

Given a partition μ into parts in

$$\{a_0 + a_1,\ a_1 + a_2,\ a_2 + a_3,\ \ldots\}$$

construct $\lambda = (\lambda_1, \lambda_2, \ldots)$ by inserting the parts of μ in nonincreasing order as follows:

To insert $a_{k-1} + a_k$ into $(\lambda_1, \lambda_2, \ldots)$:
If $k = 1$, then add a_1 to λ_1;
otherwise, if $(\lambda_1 + a_k - a_{k-1}) > c_\ell(\lambda_2 + a_{k-1} - a_{k-2})$,
add $a_k - a_{k-1}$ to λ_1, add $a_{k-1} - a_{k-2}$ to λ_2;
recursively insert $a_{k-2} + a_{k-1}$ into $(\lambda_3, \lambda_4, \ldots)$
otherwise,
add a_k to λ_1, and add a_{k-1} to λ_2.

The insertion step

To insert $a_k + a_{k-1}$ into $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots)$ either do

(i) $(\lambda_1 + a_k,\ \lambda_2 + a_{k-1},\ \lambda_3, \lambda_4, \ldots)$

or

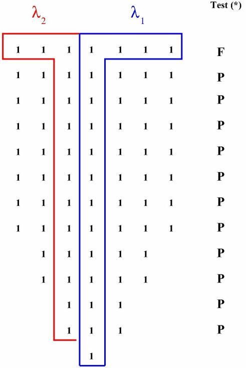
(ii) $(\lambda_1 + (a_k - a_{k-1}),\ \lambda_2 + (a_{k-1} - a_{k-2}),$

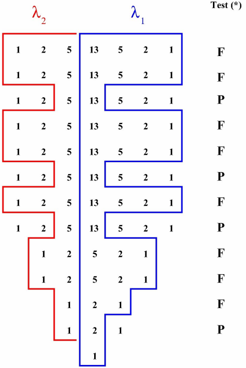
$\text{insert } (a_{k-1} + a_{k-2}) into (\lambda_3, \lambda_4, \ldots))$

How to decide?

Do (ii) if the ratio of first two parts is okay, otherwise do (i).

Sylvester Diagrams





Interpretation of $a_n^{(\ell)}$

Define $T_n^{(\ell)}$: set of ℓ-ary strings of length n that do not contain

$$(\ell-1)(\ell-2)^*(\ell-1)$$

$T_1^{(3)} = \{0, 1, 2\}$ $\qquad |T_1^{(3)}| = 3$

$T_2^{(3)} = \{00, 01, 02, 11, 10, 12, 20, 21\}$ $\qquad |T_2^{(3)}| = 8$

$T_3^{(3)}$: all 27 except 220, 221, 222, 022, 122, and 212.

$$|T_3^{(3)}| = 27 - 6 = 21$$

Theorem $\quad |\mathbf{T_n^{(\ell)}}| = \mathbf{a_{n+1}^{(\ell)}}$.

Binary numeration system

$$1\ 0\ 1\ 1\ 0 \quad \rightarrow \quad 1 * 2^4 + 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0 = 22$$

(unique representation)

Ternary numeration system

$$1\ 0\ 2\ 1\ 1 \quad \rightarrow \quad 1 * 3^4 + 0 * 3^3 + 2 * 3^2 + 1 * 3^1 + 1 * 3^0 = 138$$

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(unique representation)

A Fraenkel numeration system: ternary, but ...

$$1\ 0\ 2\ 1\ 1 \quad \rightarrow \quad 1 * 55 + 0 * 21 + 2 * 8 + 1 * 3 + 1 * 1 = 75$$

Binary numeration system

$$1\ 0\ 1\ 1\ 0 \quad \longrightarrow \quad 1 * 2^4 + 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0 = 22$$

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A Fraenkel numeration system: ternary, but ...

$$1\ 0\ 2\ 1\ 1 \quad \longrightarrow \quad 1 * 55 + 0 * 21 + 2 * 8 + 1 * 3 + 1 * 1 = 75$$

$$0\ 0\ 2\ 1\ 2 \quad \longrightarrow \quad 0 * 55 + 0 * 21 + 2 * 8 + 1 * 3 + 2 * 1 = 21$$

$$0\ 1\ 0\ 0\ 0 \quad \longrightarrow \quad 0 * 55 + 1 * 21 + 0 * 8 + 0 * 3 + 0 * 1 = 21$$

Binary numeration system

$$1\ 0\ 1\ 1\ 0 \quad \longrightarrow \quad 1*2^4+0*2^3+1*2^2+1*2^1+0*2^0=22$$

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A Fraenkel numeration system: ternary, but ...

$$1\ 0\ 2\ 1\ 1 \quad \longrightarrow \quad 1*55+0*21+2*8+1*3+1*1=75$$

$$0\ 0\ 2\ 1\ 2 \quad \longrightarrow \quad 0*55+0*21+2*8+1*3+2*1=21$$

$$0\ 1\ 0\ 0\ 0 \quad \longrightarrow \quad 0*55+1*21+0*8+0*3+0*1=21$$

but unique representation by ternary strings with no 21*2

Theorem [Fraenkel 1985] Every nonnegative integer has a unique (up to leading zeroes) representation as an ℓ-ary string which does not contain the pattern

$$(\ell - 1)\ (\ell - 2)^*\ (\ell - 1).$$

Proof Show that

$$f\ :\ \ b_n b_{n-1} \ldots b_1 \ \longrightarrow\ \sum_{i=1}^{n} b_i a_i$$

is a bijection

$$f\ :\ \ T_n^{(\ell)}\ \longrightarrow\ \{0, 1, 2, \ldots a_{n+1}^{(\ell)} - 1\}.$$

The *ℓ-representation* of an integer x is

$$[x]\ =\ f^{-1}(x).$$

λ	377	144	55	21	8	3	1	lex order
754	2	0	0	0	0	0	0	2 0 0 0 0 0 0
273		1	2	0	2	1	0	1 2 0 2 1 0 0
102			1	2	0	1	2	1 2 0 1 2 0 0
38				1	2	0	1	1 2 0 1 0 0 0
14					1	2	0	1 2 0 0 0 0 0
5						1	2	1 2 0 0 0 0 0
1							1	1 0 0 0 0 0 0

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5						1	2	1 2 0 0 0 0 0
1							1	1 0 0 0 0 0 0

Theorem [SY07]: For positive integers x, y,

$$x > c_\ell y \text{ iff } [x] \succeq [y] \cdot 0$$

$$c_3 = (3+\sqrt{5})/2 \approx 2.618$$

2.762
2.676
2.684
2.714
2.8
5

λ	377	144	55	21	8	3	1	lex order
754	2	0	0	0	0	0	0	2 0 0 0 0 0 0
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$$x > c_\ell y \quad \text{iff} \quad [x] \succeq [y]\cdot 0$$

Can revise bijection

To insert $a_k + a_{k+1}$ into $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots)$ either do

(i) $(\lambda_1 + a_k,\ \lambda_2 + a_{k-1},\ \lambda_3, \lambda_4, \ldots)$

or

(ii) $$(\lambda_1 + (a_k - a_{k-1}),\ \lambda_2 + (a_{k-1} - a_{k-2}),$$
$$\text{insert } (a_{k-1} + a_{k-2}) into (\lambda_3, \lambda_4, \ldots))$$

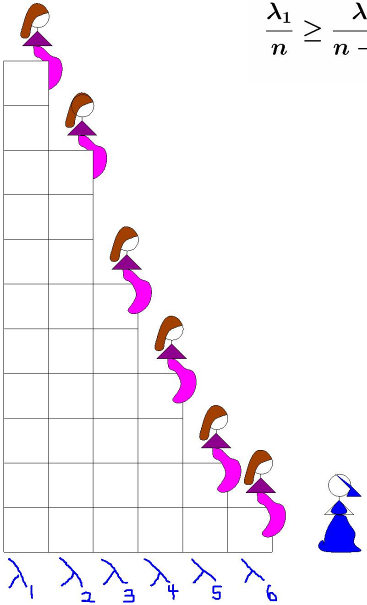
How to decide?

Do (i) if it does not create a carry in Fraenkel arithmetic; otherwise do (ii).

So, insertion becomes a 2-d form of Fraenkel arithmetic.

Lecture Hall Partitions

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0$$



The Lecture Hall Theorem [BME1] The generating function for integer sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying:

$$L_n: \quad \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0$$

is

$$L_n(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}$$

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is

$$L_n(q) = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$

$\lim_{n\to\infty}$ (Lecture Hall Theorem) = Euler's Theorem since

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_n}{1} \geq 0 \longrightarrow \text{partitions into distinct parts}$$

$$\prod_{i=1}^{n} \frac{1}{1-q^{2i-1}} \longrightarrow \text{partitions into odd parts}$$

Let $\{a_n\} = \{a_n^{(\ell)}\}$.

The ℓ-Lecture Hall Theorem [BME2]: The generating function for integer sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying:

$$L_n^{(\ell)}: \quad \frac{\lambda_1}{a_n} \geq \frac{\lambda_2}{a_{n-1}} \geq \ldots \geq \frac{\lambda_{n-1}}{a_2} \geq \frac{\lambda_n}{a_1} \geq 0$$

is

$$L_n^{(\ell)}(q) = \prod_{i=1}^{n} \frac{1}{(1 - q^{a_{i-1}+a_i})}$$

$\lim_{n\to\infty}$ (ℓ-Lecture Hall Theorem) $=$ ℓ-Euler Theorem

since as $n \to \infty$, $\quad a_n/a_{n-1} \to c_\ell$

$\Theta_n^{(\ell)}$: Bijection for the ℓ-Lecture Hall Theorem

Given a partition μ into parts in

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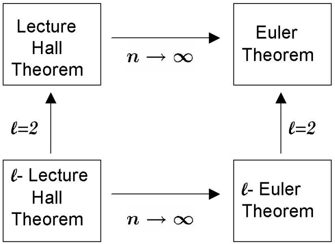
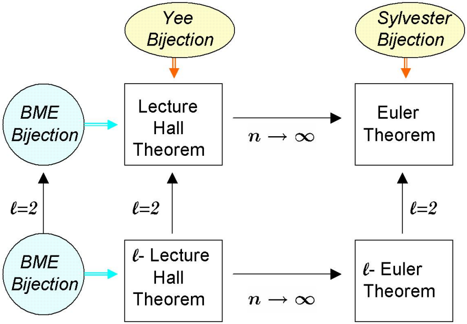
otherwise,

add a_k to λ_1, and add a_{k-1} to λ_2.

Relationship between bijections


Lecture Hall Theorem
$n \to \infty$
Euler Theorem
ℓ=2
ℓ=2
ℓ- Lecture Hall Theorem
$n \to \infty$
ℓ- Euler Theorem

Bijections



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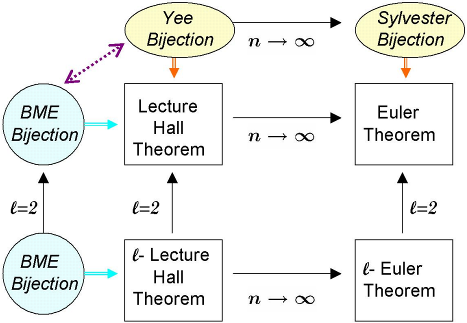
Bijections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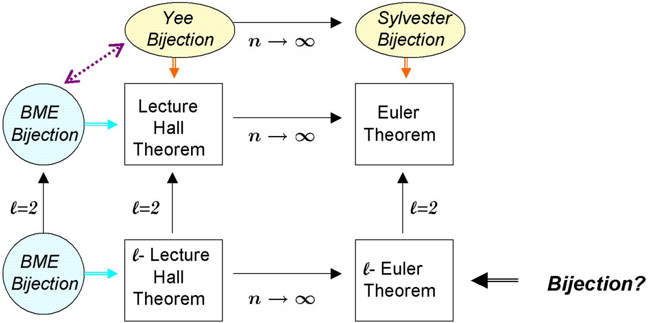
Bijections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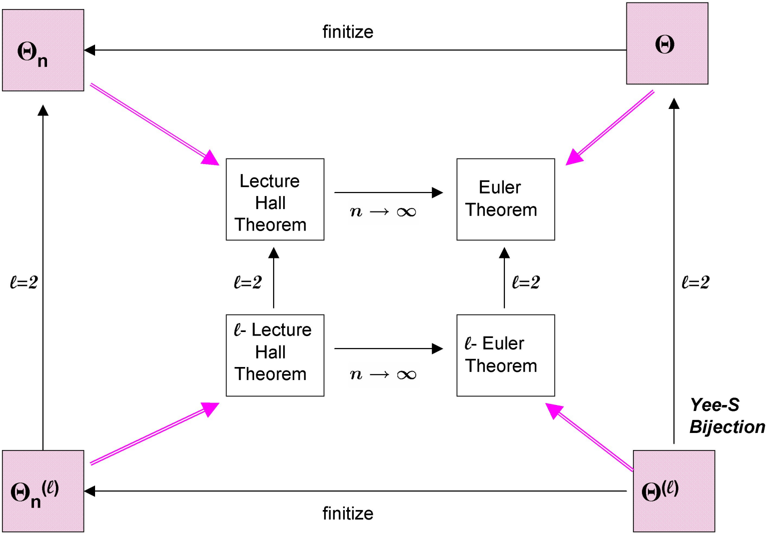
Bijection?



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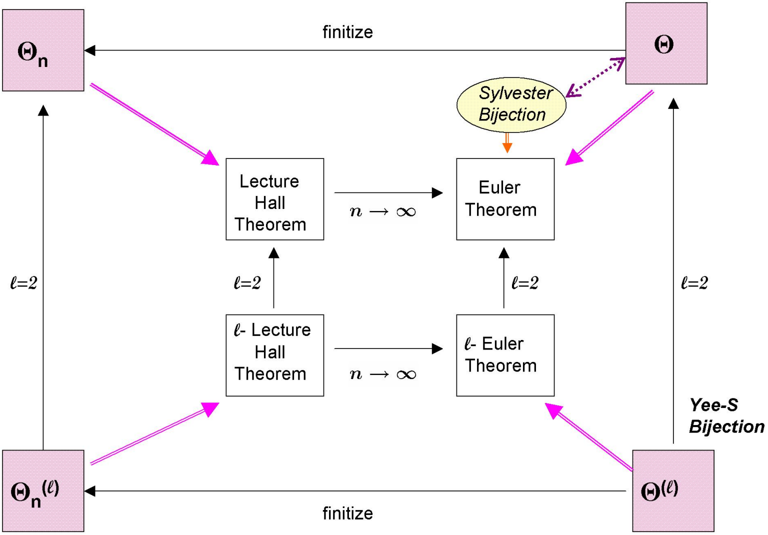

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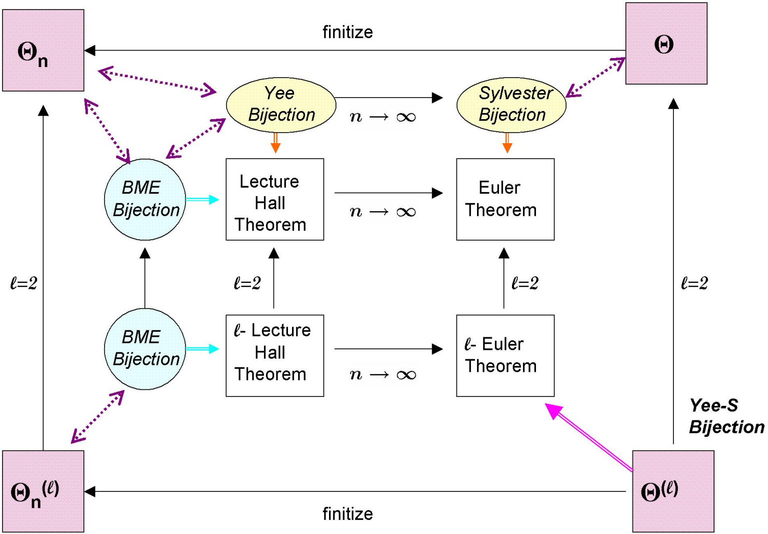

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Truncated lecture hall partitions

$$L^{(\ell)}_{n,k} \quad : \qquad \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_{k-1}}{n-k+2} \geq \frac{\lambda_k}{n-k+1} > 0$$

Theorem: [Corteel,S 2004]

$$L^{(\ell)}_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2n-k+1};q)_k},$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

Analog for ℓ-lecture hall partitions?

Truncated lecture hall partitions

$$L^{(\ell)}_{n,k,j}: \; j \geq \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_{k-1}}{n-k+2} \geq \frac{\lambda_k}{n-k+1} > 0$$

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$$L^{(\ell)}_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(q^{2n-k+1};q)_k},$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

Theorem [Corteel,Lee,S 2005]

$$L^{(\ell)}_{n,k,j}(1) = j^k \binom{n}{k}$$

Analog for ℓ-lecture hall partitions?

Theorem [Corteel,S 2004] Given positive integers $s_1, \ldots, s_n$, the generating function for the sequences $\lambda_1, \ldots, \lambda_n$ satisfying

$$\frac{\lambda_1}{s_1} \geq \frac{\lambda_2}{s_2} \geq \cdots \geq \frac{\lambda_{n-1}}{s_{n-1}} \geq \frac{\lambda_n}{s_n} \geq 0$$

is

$$\frac{\sum_{z_2=0}^{s_2-1} \sum_{z_3=0}^{s_3-1} \cdots \sum_{z_n=0}^{s_n-1} q^{\lceil \frac{s_1 z_2}{s_2} \rceil + \sum_{i=2}^{n} z_i} \prod_{i=2}^{n-1} q^{b_i \lceil \frac{z_{i+1}}{s_{i+1}} - \frac{z_i}{s_i} \rceil}}{\prod_{i=1}^{n} (1 - q^{b_i})}$$

where $b_1 = 1$ and for $2 \leq i \leq n$, $b_i = s_1 + s_2 + \cdots + s_i$.

Corollary As $q \to 1$, $(1-q)^n \times$ this gf $\longrightarrow$

$$\frac{s_2 s_3 \cdots s_n}{\prod_{i=1}^{n} b_i}.$$

Truncated ℓ-lecture hall partitions $\longrightarrow$ ℓ-nomials

Corollary For ℓ-sequence $\{a_i\}$:

$$L^{(\ell)}_{n+k,k}: \quad \frac{\lambda_1}{a_{n+k}} \geq \frac{\lambda_2}{a_{n+k-1}} \geq \frac{\lambda_k}{a_{n+1}} > 0$$

$$L^{(\ell)}_{n+k,k}(q) = ?$$

but

$$\lim_{q \to 1} \left((1-q)^k L^{(\ell)}_{n+k,k}(q)\right) = \frac{\binom{n}{k}^{(\ell)}}{(p_1 p_2 \cdots p_k)(p_{n+k} p_{n+k-1} \cdots p_{n+1})}$$

where $p_i = a_i + a_{i-1}$ and $\binom{n}{k}^{(\ell)}$ is the ℓ-nomial ...

The ℓ-nomial coefficient

$$\binom{n}{k}^{(\ell)} = \frac{a_n^{(\ell)} a_{n-1}^{(\ell)} \cdots a_{n-k+1}^{(\ell)}}{a_k^{(\ell)} a_{k-1}^{(\ell)} \cdots a_1^{(\ell)}}.$$

Example

$$\binom{9}{4}^{(3)} = \frac{2584 * 987 * 377 * 144}{21 * 8 * 3 * 1} = 174,715,376.$$

Theorem [Lucas 1878] $\binom{n}{k}^{(\ell)}$ is an integer.

like Fibonomials, e.g. Ron Knott's web page:
http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/Fibonomials.html

"Pascal-like" triangle for the ℓ-nomial coefficient:

						1						
					1		1					
				1		3		1				
			1		8		8		1			
		1		21		56		21		1		
	1		55		385		385		55		1	
1		144		2640		6930		2640		144		1

Theorem

$$\binom{n}{k}^{(\ell)} = (a_{k+1}^{(\ell)} - a_k^{(\ell)})\binom{n-1}{k}^{(\ell)} + (a_{n-k}^{(\ell)} - a_{n-k-1}^{(\ell)})\binom{n-1}{k-1}^{(\ell)}$$

Let u_ℓ and v_ℓ be the roots of the polynomial $x^2 - \ell x + 1$:

$$u_\ell = \frac{\ell + \sqrt{\ell^2 - 4}}{2}; \qquad v_\ell = \frac{\ell - \sqrt{\ell^2 - 4}}{2}$$

Then

$$u_\ell + v_\ell = \ell; \qquad u_\ell v_\ell = 1.$$

$$a_n^{(\ell)} = \frac{u_\ell^n - v_\ell^n}{u_\ell - v_\ell}.$$

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For real r, define $\Delta_r^{(\ell)} = u_\ell^r + v_\ell^r$. Then for integer n

$$\Delta_n^{(\ell)} = a_{n+1}^{(\ell)} - a_{n-1}^{(\ell)}.$$

(Δ is the ℓ analog of "2".)

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(Δ is the ℓ analog of "2".)

$$(\Delta_{n/2}^{(\ell)})^2 = u_\ell^n + 2(u_\ell v_\ell)^{n/2} + v_\ell^n = \Delta_n^{(\ell)} + 2.$$

$$\Delta_{-r}^{(\ell)} = u_\ell^{-r} + v_\ell^{-r} = v_\ell^r + u_\ell^r = \Delta_r^{(\ell)}$$

A 3-term recurrence for the ℓ-nomial [LS]

$$\binom{n}{k}^{(\ell)} = \binom{n-2}{k}^{(\ell)} + \Delta_{n-1}^{(\ell)} \binom{n-2}{k-1}^{(\ell)} + \binom{n-2}{k-2}^{(\ell)}$$

Proof: Use identities:

$$a_{n+k}^{(\ell)} - a_{n-k}^{(\ell)} = \Delta_n^{(\ell)} a_k^{(\ell)}.$$

$$a_n^{(\ell)} a_{n-1}^{(\ell)} - a_k^{(\ell)} a_{k-1}^{(\ell)} = a_{n-k}^{(\ell)} a_{n+k-1}^{(\ell)}.$$

An ℓ-nomial theorem [LS]: An analog of

$$\sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n$$

is

$$\sum_{k=0}^{n} \binom{n}{k}^{(\ell)} z^k = \prod_{i=0}^{n-1} (u_\ell^{i-(n-1)/2} + v_\ell^{i-(n-1)/2} z) =$$

$$(1+\Delta_1^{(\ell)} z + z^2)(1+\Delta_3^{(\ell)} z + z^2) \cdots (1+\Delta_{n-1}^{(\ell)} z + z^2) \quad n \text{ even}$$

$$(1+z)(1+\Delta_2^{(\ell)} z + z^2)(1+\Delta_4^{(\ell)} z + z^2) \cdots (1+\Delta_{n-1}^{(\ell)} z + z^2) \quad n \text{ odd}$$

A coin-flipping interpretation of the ℓ-nomial

For real r, let $C_r^{(\ell)}$ be a weighted coin for which the probability of tails is $u_\ell^r/\Delta_r^{(\ell)}$ and the probability of heads is $v_\ell^r/\Delta_r^{(\ell)}$.

The probability of getting exactly k heads when tossing the n coins

$$C^{(\ell)}_{-(n-1)/2}, \quad C^{(\ell)}_{1-(n-1)/2}, \quad C^{(\ell)}_{2-(n-1)/2}, \quad \ldots, \quad C^{(\ell)}_{(n-1)/2}:$$

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The probability of getting exactly k heads when tossing the n coins

$$C_{-(n-1)/2}^{(\ell)}, \quad C_{1-(n-1)/2}^{(\ell)}, \quad C_{2-(n-1)/2}^{(\ell)}, \quad \ldots, \quad C_{(n-1)/2}^{(\ell)} :$$

$$\frac{\binom{n}{k}^{(\ell)}}{\Delta_{-(n-1)/2}^{(\ell)}\Delta_{1-(n-1)/2}^{(\ell)}\cdots\Delta_{(n-1)/2-1}^{(\ell)}\Delta_{(n-1)/2}^{(\ell)}}$$

$\Delta_{j/2}^{(\ell)}$ may not be an integer, but $\Delta_{-j/2}^{(\ell)}\Delta_{j/2}^{(\ell)} = \Delta_j^{(\ell)} + 2$ is.

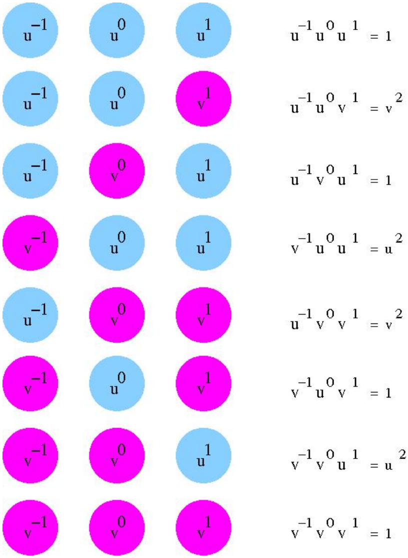


Figure: The 8 possible results of tossing the weighted coins $C_{-1}^{(\ell)}, C_0^{(\ell)}, C_1^{(\ell)}$. Heads (pink) are indicated with their "v" weight and tails (blue) with their "u" weight. The weight of each toss is shown.

Define a q-analog of the ℓ-nomial:

$$a_n^{(\ell)}(q) = \frac{u_\ell^n - v_\ell^n q^n}{u_\ell - v_\ell q}; \qquad \Delta_n^{(\ell)}(q) = u_\ell^n + v_\ell^n q^n$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q^{(\ell)} = \frac{a_n^{(\ell)}(q)\, a_{n-1}^{(\ell)}(q) \cdots a_{n-k+1}^{(\ell)}(q)}{a_k^{(\ell)}(q)\, a_{k-1}^{(\ell)}(q) \cdots a_1^{(\ell)}(q)}.$$

Then

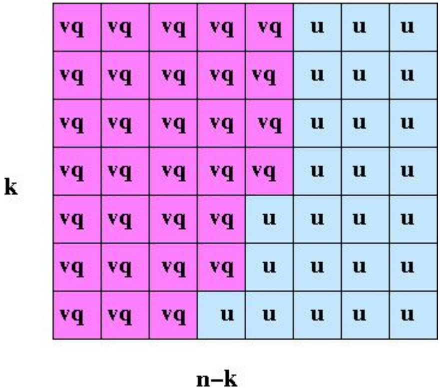
$$\left[\begin{array}{c} n \\ k \end{array}\right]_q^{(\ell)} = q^k \left[\begin{array}{c} n-2 \\ k \end{array}\right]_q^{(\ell)} + \Delta_{n-1}^{(\ell)}(q) \left[\begin{array}{c} n-2 \\ k-1 \end{array}\right]_q^{(\ell)} + q^{n-k} \left[\begin{array}{c} n-2 \\ k-2 \end{array}\right]$$

and

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q^{(\ell)} q^{k(k+1)/2} z^k = \alpha_n(q,z) \prod_{i=1}^{\lfloor n/2 \rfloor} (1 + zq^i \Delta_{n-2i+1}^{(\ell)}(q) + z^2 q^{n+1})$$

where $\alpha_n(q,z) = 1$ if n is even; $(1 + zq^{(n+1)/2})$ if n is odd.

Proofs can be derived from this partition-theoretic interpretation:



Another q-analog of the ℓ-nomial

Let $a_n(q) = (1 - q^{a_n})/(1 - q)$. Then

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q^{(\ell)} = \sum_{(\mu,f)} q^{shapeweight(\mu)} q^{fillweight(\mu,f)}$$

where the sum is over all pairs (μ, f) such μ is a partition in $[k \times (n-k)]$ and f is a filling $f(i,j)$ of the cells of $[k \times (n-k)]$ with elements of $\{0, 1, \ldots \ell - 1\}$ so that (i) no row of μ or column of μ^c contains $(\ell-1)(\ell-2)^*(\ell-1)$ and ... (a bit more)

Indexing cells of $k \times (n-k)$ bottom to top, left to right:

- cell (i,j) has a shape weight $(a_i - a_{i-1})(a_j - a_{j-1})$
- $shapeweight(\mu)$ is sum of shape weights of cells in μ
- cell (i,j) has a fill weight $a_i a_j$
- $fillweight(\mu, f)$ is $\sum_{i,j} f(i,j) a_i a_j$.

(Now starting to get something related to lecture hall partitions.)

Question: Is there an analytic proof of the ℓ-Euler theorem? When $\ell = 2$, the standard approach is to show the equivalence of the generating functions for the set of partitions into odd parts and the set of partitions into distinct parts:

$$\prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}} = \prod_{k=1}^{\infty} (1+q^k).$$

The sum/product form of Euler's theorem is

$$\prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}} = \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q;q)_k}$$

Is there an analog for the ℓ-Euler theorem? What is the sum-side generating function for partitions in which the ratio of consecutive parts is greater than c_ℓ?

Question: When $\ell = 2$, several refinements of Euler's theorem follow from Sylvester's bijection. What refinements of the ℓ-Euler theorem can be obtained from the bijection? We have some partial answers, but here's one we can't answer:

Sylvester showed that if, in his bijection, the partition μ into odd parts maps to λ (distinct parts), then the number of distinct part sizes occurring in μ is the same as the number of maximal chains in λ. (A *chain* is a sequence of consecutive integers.) Is there an analog for $\ell > 2$?

Question: Can we enumerate either of these finite sets:

- The integer sequences $\lambda_1, \ldots, \lambda_n$ in which the ratio of consecutive parts is greater than c_ℓ and $\lambda_1 < a_{n+1}^{(\ell)}$?
  We have a combinatorial characterization: these can be viewed as fillings of a staircase of shape $(n, n-1, \ldots, 1)$ such that (i) the filling of each row is an ℓ-ary string with no $(\ell-1)(\ell-2)^*(\ell-1)$ and (ii) the rows are weakly decreasing, lexicographically. For $\ell = 2$ the answer is 2^n.
- The integer sequences $\lambda_1, \ldots, \lambda_n$ satisfying

$$1 \geq \frac{\lambda_1}{a_n} \geq \frac{\lambda_2}{a_{n-1}} \geq \ldots \geq \frac{\lambda_{n-1}}{a_2} \geq \frac{\lambda_n}{a_1} \geq 0$$

  For $\ell = 2$ this is the same set as above and the answer is 2^n.

Question: What is the generating function for truncated ℓ-lecture hall partitions?

Question: What is the right q-analog of the ℓ-nomial (and the right interpretation) to explain and generalize lecture hall partitions?
(We know what lecture hall partitions look like in the "ℓ-world". What about ordinary partitions? partitions into distinct parts?)

Question: There are several q-series identities related to Euler's theorem, such as Lebesgue's identity the Roger's-Fine identity Cauchy's identity Are there ℓ-analogs?

Sincere thanks to ...

You, the audience

The FPSAC08 Organizing Committee

The FPSAC08 Program Committee

Luc Lapointe

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North Carolina State University

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Some of the special sessions:

- Applications of Algebraic and Geometric Combinatorics (Sullivant, Savage)
- Rings, Algebras, and Varieties in Combinatorics (Hersh, Lenart, Reading)
- Recent Advances in Symbolic Algebra and Analysis (Singer, Szanto)
- Kac-Moody Algebras, Vertex Algebras, Quantum Groups, and Applications (Bakalov, Misra, Jing)
- Enumerative Geometry and Related Topics (Rimayi, Mihalcea)
- Homotopical Algebra with Applications to Mathematical Physics (Lada, Stasheff)
- Low Dimensional Topology and Geometry (Dunfield, Etnyre, Ng)